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Probability Operator Measure and Phase Measurement in a Deformed Hilbert Space

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We discuss the probability operator measure and phase measurement in a deformed Hilbert space.

1. INTRODUCTION

We consider the set

$$H_q = \{f: f(z) = \Sigma\, a_n z^n \text{ where } \Sigma[n]!\ |a_n|^2 < \infty\}$$

where $[n] = (1 - q^n)/(1 - q)$, $0 < q < 1$.

For $f, g \in H_q$, $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$, and $g(z) = \Sigma_{n=0}^{\infty} b_n z^n$ we define addition and scalar multiplication as follows:

$$f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n \tag{1}$$

and

$$\lambda \circ f(z) = \sum_{n=0}^{\infty} \lambda a_n z^n \tag{2}$$

It is easily seen that H_q forms a vector space with respect to usual pointwise scalar multiplication and pointwise addition by (1) and (2). We observe that $e_q(z) = \Sigma_{n=0}^{\infty} (z^n/[n]!)$ belongs to H_q.

Now we define the inner product of two functions $f(z) = \Sigma\, a_n z^n$ and $g(z) = \Sigma\, b_n z^n$ belonging to H_q as

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$$(f, g) = \sum [n]!\, \bar{a}_n b_n \tag{3}$$

The corresponding norm is given by

$$\|f\|^2 = (f, f) = \sum [n]!\, |a_n|^2 < \infty$$

With this norm derived from the inner product it can be shown that H_q is a complete normed space. Hence H_q forms a Hilbert space.

In a recent paper [1] we have proved that the set $\{f_n \equiv z^n/\sqrt{[n]!},\ n = 0, 1, 2, 3, \ldots\}$ forms a complete orthonormal set. If we consider the following actions on H_q:

$$\begin{aligned} Tf_n &= \sqrt{[n]}\, f_{n-1} \\ T^* f_n &= \sqrt{[n+1]}\, f_{n+1} \end{aligned} \tag{4}$$

where T is the backward shift and its adjoint T^* is the forward shift operator on H_q, then we have shown [1] that the solution of the eigenvalue equation

$$Tf = \alpha f \tag{5}$$

is given by

$$f_\alpha = e_q(|\alpha|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}}\, f_n \tag{6}$$

We call f_α a *coherent vector* in H_q.

The paper is divided into five sections. In this section we have given an introduction stating coherent vectors in H_q. In Section 2 we describe the probability operator measure (POM) in H_q. In Section 3 we discuss phase distribution in H_q. In Section 4 we study the phase estimation problem, and in Section 5 we give a conclusion.

2. PROBABILITY OPERATOR MEASURE

A discrete spectrum *probability operator measure (POM)* consists of a set of Hermitian, positive-semidefinite operators $\{\Pi_n\colon n \in \mathbb{N}\}$ which resolves the identity

$$I = \sum_{n \in \mathbb{N}} \Pi_n \tag{7}$$

Measurement of this POM, by definition, gives a discrete, classical random variable with probability distribution

$$P(n, g) = (g, \Pi_n g) \qquad \text{for} \quad n \in \mathbb{N} \tag{8}$$

where g is any vector of unit norm in H_q.

In order that the laws of classical probability be satisfied, it is necessary and sufficient that

$$0 \le P(n, g) \le 1, \qquad \sum_{n=0}^{\infty} P(n, g) = 1 \tag{9}$$

are satisfied for arbitrary g of unit norm in H_q.

We know that the sequence $f_n = z^n/\sqrt{[n]!}$ forms a complete orthonormal sequence in H_q and comprises eigenvectors of the operator $N = T^*T$ such that

$$Nf_n = [n] f_n \tag{10}$$

Measurement of N for any arbitrary vector $g \in H_q$ of unit norm yields a discrete-valued classical random variable with probability distribution

$$P(f_n, g) = |(f_n, g)|^2 \qquad \text{for} \quad n = 0, 1, 2, \ldots \tag{11}$$

In order that the law of classical probability be satisfied, it is necessary and sufficient that

$$0 \le P(f_n, g) \le 1, \qquad \sum_{n=0}^{\infty} P(f_n, g) = 1 \tag{12}$$

for arbitrary $g \in H_q$ of unit norm.

The completeness of $\{f_n\}$ guarantees that the prescription in equation (11) obeys equation (12). For, if we expand the arbitrary vector g of unit norm in terms of f_n, we have

$$g = \sum_{n=0}^{\infty} (f_n, g) f_n \tag{13}$$

$$= \sum_{n=0}^{\infty} |f_n\rangle\langle f_n| g$$

Where we define the operator

$$|f_n\rangle\langle f_n|: \quad H_q \to H_q$$

by

$$|f_n\rangle\langle f_n| = (f_n, g) f_n$$

Equation (12) is now easily verified from equation (11) and equation (13).

Thus, N operator measurement is equivalent to the POM

$$\{\Pi_n = |f_n\rangle\langle f_n| : n = 0, 1, 2, \ldots\} \tag{14}$$

Similarly, a continuous spectrum POM consists of a set of Hermitian, positive-semidefinite differential operators $\{d\Pi(\beta)\colon \beta \in \mathbb{C}\}$ which resolve the identity

$$I = \int_{\beta \in \mathbb{C}} d\Pi(\beta) \tag{15}$$

The result of measuring this POM is, by definition, a continuous classical random variable whose probability density function is given by

$$p(\beta, g) = \frac{(g, d\Pi(\beta)g)}{d\beta} \qquad \text{for} \quad \beta \in \mathbb{C} \tag{16}$$

where g is any vector of unit norm in H_q.

We know that the backwardshift T has eigenvectors—the coherent vectors f_α (6). These vectors are not orthogonal, but they form a resolution of the identity

$$I = \frac{1}{2\pi} \int_{\alpha \in \mathbb{C}} d\mu(\alpha)\, |f_\alpha\rangle\langle f_\alpha| \tag{17}$$

where

$$d\mu(\alpha) = e_q(|\alpha|^2)e_q(-|\alpha|^2)d_q|\alpha|^2\, d\theta$$

with $\alpha = re^{i\theta}$, which defines a T-POM

$$d\Pi(\alpha) \equiv |f_\alpha\rangle\langle f_\alpha| \frac{d\mu(\alpha)}{2\pi} \qquad \text{for} \quad \alpha \in \mathbb{C} \tag{18}$$

The outcome of the T-POM is a complex-valued continuous classical random variable with probability density function

$$p(\alpha, g) = \frac{(g, d\Pi(\alpha)g)}{d\mu(\alpha)} = \frac{1}{2\pi}|(f_\alpha, g)|^2 \qquad \text{for} \quad \alpha \in \mathbb{C} \tag{19}$$

where g is any vector of unit norm in H_q.

Because of (17), it follows that

$$p(\alpha, g) \geq 0, \qquad \int_{\alpha \in \mathbb{C}} d\mu(\alpha)p(\alpha, g) = 1 \tag{20}$$

hold for any vector g of unit norm in H_q.

3. PHASE DISTRIBUTION

To obtain the phase distribution we consider first the *phase operator* $P = (q^n + T^*T)^{-1/2}T$ and try to find the solution of the following eigenvalue equation:

$$Pf_\beta = \beta f_\beta \tag{21}$$

We arrive at

$$\begin{aligned} f_\beta &= \sum_{n=0}^{\infty} a_n \sqrt{[n]!}\, f_n \\ &= a_0 \sum_{n=0}^{\infty} \beta^n \sqrt{\frac{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^n + [n-1])}{[n]!}}\, f_n \end{aligned}$$

where $\beta = |\beta| e^{i\theta}$ is a complex number. These vectors are normalizable in a strict sense only for $|\beta| < 1$.

For details of the calculations see ref. 2.

Now, if we take $a_0 = 1$ and $|\beta| = 1$ we have

$$f_\beta = \sum_{n=0}^{\infty} e^{in\theta} \sqrt{\frac{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^n + [n-1])}{[n]!}}\, f_n \tag{22}$$

Henceforth, we shall denote this vector as

$$f_\theta = \sum_{n=0}^{\infty} e^{in\theta} \sqrt{\frac{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^n + [n-1])}{[n]!}}\, f_n \tag{23}$$

$0 \le \theta \le 2\pi$, and call f_θ a *phase vector* in H_q.

The phase vectors f_θ are neither normalizable nor orthogonal. The completeness relation

$$I = \frac{1}{2\pi} \int_X \int_0^{2\pi} d\nu(x, \theta) | f_\theta \rangle \langle f_\theta | \tag{24}$$

where

$$d\nu(x, \theta) = d\mu(x)\, d\theta \tag{25}$$

may be proved as follows:

Here we consider the set X consisting of the points $x = 0, 1, 2, \ldots,$ and $\mu(x)$ is the measure on X which equals

$$\mu_n \equiv \frac{[n]!}{(q + [0])(q^2 + [1]) \ldots (q^n + [n-1])}$$

at the point $x = n$; θ is the Lebesgue measure on the circle.

Define the operator

$$| f_\theta \rangle \langle f_\theta | : \quad H_q \to H_q \tag{26}$$

by

$$|f_\theta\rangle\langle f_\theta| f = (f_\theta, f) f_\theta \tag{27}$$

with $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Now,

$$\begin{aligned}(f_\theta, f) &= \sum_{n=0}^{\infty} [n]! \frac{e^{-in\theta}}{\sqrt{[n]!}} \sqrt{\frac{(q+[0])(q^2+[1])(q^3+[2])\ldots(q^n+[n-1])}{[n]!}}\, a_n \\ &= \sum_{n=0}^{\infty} e^{-in\theta} \sqrt{(q+[0])(q^2+[1])(q^3+[2])\ldots(q^n+[n-1])}\, a_n \end{aligned} \tag{28}$$

Then,

$$\begin{aligned}(f_\theta, f) f_\theta &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n e^{i(m-n)\theta} \sqrt{\frac{(q+[0])(q^2+[1])\ldots(q^m+[m-1])}{[m]!}} \\ &\quad\times \sqrt{(q+[0])(q^2+[1])\ldots(q^n+[n-1])}\, f_m \end{aligned} \tag{29}$$

Using

$$\int_0^{2\pi} d\theta\, e^{i(m-n)\theta} = 2\pi\delta_{mn} \tag{30}$$

we have

$$\begin{aligned}&\frac{1}{2\pi}\int_X \int_0^{2\pi} d\nu(x,\theta) |f_\theta\rangle\langle f_\theta| f \\ &= \int_X d\mu(x) \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n f_m \sqrt{\frac{(q+[0])(q^2+[1])\ldots(q^m+[m-1])}{[m]!}} \\ &\quad\times \sqrt{(q+[0])(q^2+[1])\ldots(q^n+[n-1])}\, \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\theta}\, d\theta \\ &= \sum_{n=0}^{\infty} a_n f_n \int_X \frac{(q+[0])(q^2+[1])\ldots(q^n+[n-1])}{\sqrt{[n]!}}\, d\mu(x) \\ &= \sum_{n=0}^{\infty} a_n f_n \frac{(q+[0])(q^2+[1])\ldots(q^n+[n-1])}{\sqrt{[n]!}} \\ &\quad\times \frac{[n]!}{(q+[0])(q^2+[1])\ldots(q^n+[n-1])}\end{aligned}$$

$$= \sum_{n=0}^{\infty} \sqrt{[n]!}\, a_n f_n$$

$$= f \tag{31}$$

Thus, (24) follows.

The *phase distribution* over the window $0 \leq \theta \leq 2\pi$ for any vector f is then defined by

$$P(\theta) = \frac{1}{2\pi} |(f_\theta, f)|^2 \tag{32}$$

4. PHASE ESTIMATION

Once we have the POM information, we are ready to discuss the phase estimation problem. Without loss of generality, we assume that $0 \leq \theta \leq 2\pi$. The class of POMs we must optimize over in order to find the best phase estimate is taken to be

$$\{d\hat{\Pi}(\theta): 0 \leq \theta \leq 2\pi\}$$

where

$$d\hat{\Pi}(\theta) = d[\hat{\Pi}(\theta)]^\dagger \qquad \text{and} \qquad I = \int_0^{2\pi} d\hat{\Pi}(\theta) \tag{33}$$

The conditional probability density, given the phase operator

$$P = (q^n + T^*T)^{-1/2}T$$

for obtaining a phase value θ from this POM is

$$p(\theta, P) = \frac{(g, d\hat{\Pi}(\theta)g)}{d\nu(x, \theta)} \qquad \text{for} \quad 0 \leq \theta \leq 2\pi, \quad x \text{ an integer} \tag{34}$$

where g is a vector of unit norm in H_q.

We choose the POM $d\hat{\Pi}(\theta)$ and the input vector g to optimize our estimate of the phase shift P. For a given POM and the input vector, equation (34) supplies the PDF needed to perform a classical maximal likelihood estimation. The observed phase value θ is our estimate of P. In order for this estimate to be one of maximum likelihood, we restrict our attention to the POMs satisfying

$$P_{ML}(\theta) = arg \max_{\theta} p(\theta, P) \qquad \text{for} \quad 0 \leq \theta \leq 2\pi \tag{35}$$

and optimize our estimate over $d\hat{\Pi}$ and g by maximizing the peak likelihood, minimizing $\delta\theta \equiv 1/p(\theta, P)$.

For the input vector

$$g = \sum_{n=0}^{\infty} (f_n, g) f_n$$

where

$$(f_n, g) = |(f_n, g)| e^{ik_n}, \qquad n = 0, 1, 2, \ldots \tag{36}$$

$\delta\theta$ is minimized by the following POM:

$$d\hat{\Pi}(\theta) = |f_\theta^g\rangle\langle f_\theta^g| \frac{d\nu(x, \theta)}{2\pi} \tag{37}$$

where

$$d\nu(x, \theta) = d\mu(x)\, d\theta, \qquad 0 \leq \theta \leq 2\pi$$

as in (25) and

$$f_\theta^g \equiv \sum_{n=0}^{\infty} e^{in\theta + ik_n} \times \sqrt{\frac{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^n + [n-1])}{[n]!}}\, f_n \tag{38}$$

To calculate the reciprocal peak likelihood $\delta\theta$ with this optimum POM to estimate P we observe first

$$p(\theta, P) = \frac{(g, d\hat{\Pi}(\theta) g)}{d\nu(x, \theta)} = \frac{|(f_\theta^g, g)|^2}{2\pi} = \frac{1}{2\pi} \left| \sum_{n=0}^{\infty} e^{-in\theta} \sqrt{\frac{(q + [0])(q^2 + [1]) \ldots (q^n + [n-1])}{[n]!}}\, |(f_n, g)| \right|^2 \tag{39}$$

Hence a suitable peak likelihood $\delta\theta$ for maximum $p(\theta, P)$ can be [4]

$$\delta\theta = 2\pi |(f_\theta^g, g)|^{-2} = 2\pi \left| \sum_{n=0}^{\infty} \sqrt{\frac{(q + [0])(q^2 + [1]) \ldots (q^n + [n-1])}{[n]!}}\, |(f_n, g)| \right|^{-2} \tag{40}$$

which is independent of the phases $\{k_n\}$. In fact, $p(\theta, P)$ is independent of the phases $\{k_n\}$.

As the peak likelihood $\delta\theta$ is independent of $\{k_n\}$, we can assume, without loss of generality, that the input vector $g = \sum_{n=0}^{\infty} (f_n, g) f_n$ has positive real coefficient (f_n, g). Equation (38) then reduces to

$$f_\theta^g = f_\theta = \sum_{n=0}^{\infty} e^{in\theta} \sqrt{\frac{(q + [0]) \ldots (q^n + [n-1])}{[n]!}} f_n \tag{41}$$

for $0 \le \theta \le 2\pi$, which is the solution of the eigenvalue equation (21),

$$Pf_\theta = e^{i\theta} f_\theta$$

Now we consider the operator

$$U = \sum_{n=0}^{\infty} e^{-ik_n} | f_n \rangle \langle f_n | \tag{42}$$

Observe that

$$UU^* = U^*U = I$$

Thus, U is a unitary transformation.

Now, for an arbitrary input vector g the optimum POM from equation (37) is equivalent to performing the unitary transformation U followed by the POM

$$d\,\Pi(\theta) = | f_\theta \rangle \langle f_\theta | \frac{d\nu(x, \theta)}{2\pi} \tag{43}$$

where

$$d\nu(x, \theta) = d\mu(x)\, d\theta, \qquad 0 \le \theta \le 2\pi$$

as in (24) and (25), for

$$\begin{aligned} Uf_\theta^g &= \sum_{n=0}^{\infty} e^{in\theta + ik_n} e^{-ik_n} \sqrt{\frac{(q + [0]) \ldots (q^n + [n-1])}{[n]!}} f_n \\ &= \sum_{n=0}^{\infty} e^{in\theta} \sqrt{\frac{(q + [0]) \ldots (q^n + [n-1])}{[n]!}} f_n \\ &= f_\theta \end{aligned} \tag{44}$$

where f_θ^g is given by (38).

Shifting the input vector's phase by the phase operator P amounts to

$$(f_n, g) \to e^{in\theta_0}(f_n, g) \qquad \text{for} \quad n = 0, 1, 2, \ldots \tag{45}$$

By rotating out the input phases k_n with the U transformation we get the transformed input as

$$e^{in\theta_0}(f_n, g) \xrightarrow{U} e^{in\theta_0} |(f_n, g)| \tag{46}$$

The effect of the POM on equation (43) on this transformed vector

$$g' = \sum_{n=0}^{\infty} e^{in\theta_0} |(f_n, g)| \, f_n \tag{47}$$

gives the classical phase with PDF

$$\begin{aligned} p(\theta, P) &= \frac{(g', d\Pi(\theta)g')}{d\nu(x, \theta)} \\ &= \frac{1}{2\pi} (g', |f_\theta\rangle\langle f_\theta| g') \\ &= \frac{|(f_\theta, g'|^2}{2\pi} \\ &= \frac{1}{2\pi} \left| \sum_{n=0}^{\infty} e^{in(\theta_0 - \theta)} \sqrt{\frac{(q + [0]) \ldots (q^n + [n-1])}{[n]!}} \, |(f_n, g)| \right|^2 \end{aligned} \tag{48}$$

From the above equation it is clear that ML estimate obeys $P_{ML}(\theta) = \theta$.

Thus, the POM in equation (24) leads to the ML phase estimate for all vectors in H_q. Thus, to achieve our goal of jointly optimizing phase estimate performance over both the measurement and the input vector, it remains for us to minimize $\delta\theta$ from equation (40) by appropriate choice of input vector. Specifically, the coefficients $\{(f_n, g)\}$ for the input vector must minimize the right side of equation (40) subject to the normalization constraint

$$\sum_{n=0}^{\infty} |(f_n, g)|^2 = 1 \tag{49}$$

and the average number constraint

$$\sum_{n=0}^{\infty} [n] |(f_n, g)|^2 = N_0 \tag{50}$$

where $N_0 = (g, T^*Tg)$.

Without loss of generality, we shall assume that (f_n, g) are positive real. Now, maximize

$$\begin{aligned} L(g, \lambda_1, \lambda_2) \equiv \frac{1}{2\pi} \left[\sum_{n=0}^{\infty} (f_n, g) \right]^2 &+ \lambda_1 \left[\sum_{n=0}^{\infty} (f_n, g)^2 - 1 \right] \\ &+ \lambda_2 \left[\sum_{n=0}^{\infty} [n](f_n, g)^2 - N_0 \right] \end{aligned} \tag{51}$$

where λ_1 and λ_2 are Lagrange multipliers.

It is straightforward to show that

$$(f_n, g) = \frac{c}{k + [n]} \qquad \text{for} \quad n = 0, 1, 2, \ldots \tag{52}$$

achieves the required stationary point for L, where c and k are positive constants depending on the Lagrange multipliers. For brevity we shall chose $k = 1$.

As we know that $[n] \geq n$ for $q > 0$, we have

$$\frac{c/(1 + [n])}{1/n} \leq \frac{c}{1/n + 1}$$

Hence we see that

$$\lim_{n\to\infty} \frac{c/(1 + [n])}{1/n} \leq c$$

Thus, the series $\Sigma_{n=0}^{\infty} [c/(1 + [n])]$ and $\Sigma_{n=0}^{\infty} (1/n)$ converge or diverge together. But $\Sigma_{n=0}^{\infty} (1/n)$ diverges. Hence, we must introduce a truncation parameter in equation (52). That is, we have

$$\begin{aligned} (f_n, g) &= \frac{c}{1 + [n]} \qquad \text{for} \quad n = 0, 1, 2, \ldots, s \\ &= 0 \qquad \text{for} \quad n > s \end{aligned} \tag{53}$$

Now, we have

$$\begin{aligned} N_0 &= \sum_{n=0}^{s} [n] \cdot |(f_n, g(\alpha))|^2 \\ &= \sum_{n=0}^{s} [n] \frac{c^2}{(1 + [n])^2} \\ &= \sum_{n=0}^{s} \frac{c^2}{1 + [n]} - 1 \end{aligned} \tag{54}$$

where we have used equations (50), (51), and (53) with the truncation point s. Then,

$$\begin{aligned} \delta\theta &= 2\pi \left[\sum_{n=0}^{s} \sqrt{\frac{(q + [0])(q^2 + [1]) \ldots (q^n + [n - 1])}{[n]!}} (f_n, g) \right]^{-2} \\ &= 2\pi c^2 \left[\sum_{n=0}^{s} \sqrt{\frac{(q + [0])(q^2 + [1]) \ldots (q^n + [n - 1])}{[n]!}} \frac{c^2}{1 + [n]} \right]^{-2} \end{aligned}$$

$$< 2\pi c^2 A\left[\sum_{n=0}^{s} \frac{c^2}{1 + [n]}\right]^{-2} = \frac{2\pi c^2 A}{(N_0 + 1)^2} \approx \frac{2\pi c^2 A}{N_0^2} \tag{55}$$

for $N_0 \gg 1$. Here A is a constant.

5. CONCLUSION

We know [3] that ML phase estimation with the optimized state leads to $\delta\theta \sim 1/N_0^2$ for the reciprocal peak likelihood performance, where we are interested in the behavior at high average photon number, namely $N_0 \gg 1$. In this paper we show that in the deformed case $\delta\theta$ can be even less than $1/N_0^2$.

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